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CONTACT PROBLEMS ON THE FORCED STATIONARY VIBRATIONS OF BEAMS ON AN ELASTIC STRIP, HALF-STRIP, AND RECTANGLE

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Dynamic problems of elasticity theory are considered for a strip, a half-strip, and a rectangle partially reinforced by constant-stiffness beams. Questions of wave propagation, energy transfer, stress concentration under the beam ends are studied. The possibility of extending the solutions obtained to other classes of mixed problems is discussed.

The solution of the inhomogeneous problem of stationary vibrations of a semi-infinite beam on a strip, and the system of piecewise-homogeneous solutions of this problem are constructed in quadratures. Analogous problems for a half-strip and rectangle, particularly periodic problems and the problem of the dynamics of one finite beam on a strip, are reduced to normal Poincare–Koch systems whose matrix elements decrease exponentially in the numbers of rows and columns, by the use of new generalized orthogonality relationships with a load.

The method of piecewise-homogeneous solutions was used earlier only in elastostatic problems [1 — 9], the problems under consideration were also not solved by other methods.

1. Let a semi-infinite beam $x \geqslant 0$, $y = 1$ with constant stiffness D and linear mass μ lie on an elastic strip $-\infty < x < \infty$, $0 \leqslant y \leqslant 1$. There is no friction between the beam and the strip, and the foundation of the strip is in a sliding frame. Stationary normal loads on the free part of the strip and beam, as well as the moment and transverse force at the point $x = 0$, $y = 1$ act in phase, and equal to $f_1(x) \cos \omega t$, $f_2(x) \cos \omega t$, $P_2 \cos \omega t$, $P_3 \cos \omega t$, respectively, where t is the time, and ω is the frequency of the forced vibrations. The loads $f_1(x)$, $f_2(x)$ are local or decrease exponentially as $|x| \to \infty$, at $t = 0$ the moment P_2 is clockwise, and the force P_3 is along the y axis. The conditions at infinity correspond to the Mandelshtam principle: the energy flux averaged with respect to the strip section and the period $T = 2\pi\omega^{-1}$ for each wave being propagated is directed towards $\pm\infty$ as $x \to \pm\infty$ [10]. The local strain energy of the strip is constrained in the neighborhood of the point $x = 0$, $y = 1$.

We seek the solution of the boundary value problem for the strip $\mathbf{u}_* = Re\,[\mathbf{u}(x, y)\, e^{i\omega t}]$ in the Papkovich — Neuber form [11]

$$2G\mathbf{u} = 4(1-\nu)\boldsymbol{\Phi} - \operatorname{grad} F, \quad \boldsymbol{\Phi} \equiv (\Phi_1, \Phi_2), \quad \Phi_1 \equiv 0 \tag{1.1}$$
$$F = \Phi_0 + x\Phi_1 + y\Phi_2, \quad \Delta\Phi_0 + k_1^2\Phi_0 = (k_2^2 - k_1^2)\, y\, \Phi_2$$
$$\Delta\Phi_2 + k_2^2\Phi_2 = 0, \quad k_1^2 = k_2^2(1-2\nu)(2-2\nu)^{-1}, \quad k_2^2 = \rho\omega^2 G^{-1}$$

where ρ is the density of the strip material, ν and G are its elastic constants. Let u_1 and u_2 be projections of the vector $\mathbf{u}$ on the x and y axes, and u_3, u_4, u_5 the stress tensor components $\tau_{xy}, \sigma_x, \sigma_y$, respectively.

Let us write down the boundary conditions for the complex amplitudes of the displacements and stresses

$$u_2(x, 0) = u_3(x, 0) = u_3(x, 1) = 0 \tag{1.2}$$

$$u_5(x, 1) = f_1(x) \quad (x < 0), \quad \eta(x) \equiv D\partial^4 u_2 / \partial x^4 - \alpha u_2 + u_5 = f(x) \quad (x \geqslant 0,\ y = 1) \tag{1.3}$$

$$\psi_m(0) = P_m \quad (m = 2, 3), \quad \psi_m(x) \equiv D\partial^m u_2 / \partial x^m, \quad y = 1 \tag{1.4}$$

$$(D = E_0 h^3 [12(1 - \nu_0^2)]^{-1}, \ \alpha = \mu h \omega^2)$$

Here E_0 and ν_0 are elastic constants, and h is the beam thickness.

Applying a bilateral Laplace transform to equations (1.1), we obtain from the conditions (1.2)

$$u_s(x, y) = \frac{1}{2\pi i} \int_L C(p) U_s(p, y) e^{px} dp, \quad s = 1, 2, \ldots, 5 \tag{1.5}$$

$$2GU_1(p, y) = pq_2(q^2 \sin q_2 \cos q_1 y - q_1 q_2 \sin q_1 \cos q_2 y)$$

$$2GU_2(p, y) = q_1 q_2 (q^2 \sin q_2 \sin q_1 y - p^2 \sin q_1 \sin q_2 y)$$

$$q^2 = p^2 + {}^1/_2 k_2^2, \quad q_m^2 = p^2 + k_m^2, \quad m = 1, 2$$

Substituting (1.5) into (1.3), we obtain two equations

$$\sigma^+(p) + \sigma^-(p) = C(p) N_1(p), \quad \eta^+(p) + \eta^-(p) = C(p) N_2(p), \quad p \in L \tag{1.6}$$

$$\sigma^\pm(p) = \pm \int_0^{\pm\infty} u_5(x, 1) e^{-px} dx, \quad \eta^\pm(p) = \pm \int_0^{\pm\infty} \eta(x) e^{-px} dx$$

$$N_1(p) = U_5(p, 1) = q^4 q_2 \sin q_2 \cos q_1 - p^2 q_1 q_2^2 \sin q_1 \cos q_2$$

$$N_2(p) = N_1(p) + (Dp^4 - \alpha) U_2(p, 1), \quad 2GU_2(p, 1) = k_2^2 q_1 q_2 \sin q_1 \sin q_2$$

Here $N_r(p)$ $(r = 1, 2)$ are even entire functions having a countable set of zeroes a_{kr}. After expanding the integrals (1.5) in a residue series (at the complex poles $p = a_{kr}$ each residue is a wave damped along x, the homogeneous solution), the pure imaginary poles whose number S_r is finite and dependent on ω, determine the waves being propagated. As the frequency ω increases, the majority of these zeroes arrive at the imaginary axis through the origin from the complex plane. The cases $a_{kr} = 0$ can hold only for critical values of the frequencies ω which agree with the natural vibrations frequencies of the strip cross-section. Hence, the number of imaginary zeroes $S_r(\omega)$ for a given frequency ω is determined by the number of critical frequencies not exceeding ω, i.e.,

$$S_r(\omega) = \left[\frac{\omega}{\pi c_1}\right] + \left[\frac{\omega}{\pi c_2}\right] + 1 \quad c_m = \frac{\omega}{k_m}$$

where $[a]$ is the integer part of the number a, and c_1 and c_2 are the propagation velocities of the tension and distortion waves referred to the dimensional width of the strip. In principle, this relationship can be spoiled in individual frequency bands because of the zeroes arriving at the imaginary axis not through the origin. However, such cases are the exception [12] and change the quantity S_r insignificantly. For certain values of ω the functions $N_r(p)$ have multiple zeroes, and these cases are not examined here. We shall number the pure imaginary zeroes a_{kr}, which differ in aboslute value, from one to $S_r > 0$, while the numbering of the complex (and real) zeroes a_{kr} in the half-plane $\operatorname{Re} p > 0$ starts with $S_r + 1$; the zeroes symmetric to those mentioned with respect to the origin will be denoted by a_{-kr}, $a_{-kr} = -a_{kr}$. Relying on the Rouchét theorem and following [13], it can be proved that all $|a_{kr}| \gg \omega$ greater then zero in absolute value are complex. For $\operatorname{Re} p > 0$, $\operatorname{Im} p > 0$, they are defined by the asymptotic formulas

$$a_{k1} = (1/{}_2k - l_1)\pi - 1/{}_4\pi + 1/{}_2 i \ln k\pi + iO(1) \qquad (1.7)$$
$$a_{k2} = (1/{}_2k - l_2)\pi + iO(1)$$

where $l_r \geqslant 0$ are integers dependent on ω, k to even. The larger zeroes a_{kr} have odd numbers in the quadrant $\operatorname{Re} p > 0$, $\operatorname{Im} p < 0$.

We select the contour L and the pure imaginary zeroes with positive indices k so that the Mandel'shtam principle would be satisfied and the numbering of the zeroes in the last expressions would be natural, from one to S_r. Let $p = i\gamma$ be a point of the imaginary axis, $c_{kr} = d\omega / d\gamma$ for $i\gamma = a_{kr}$ the group velocity of the kr-th wave being propagated from the left $(r = 1)$ or the right $(r = 2)$ to infinity, Q_{kr} the energy density of this wave averaged over the period T, P_{kr} its flux averaged over T and over the section $x =$ const. Let us use the Reynolds — Rayleigh formula $P_{kr} = c_{kr} Q_{kr}$ (see [14], p. 239; M. A. Leontovich [15] and Lighthill [16] obtained the most general proof of the validity of this formula). Since $N_r(-p) = N_r(p)$, then $c_{-kr} = -c_{kr}$, and since $Q_{kr} > 0$, the flux $P_{kr} < 0$ corresponds to one out of every two eigenvalues with modulus equal to $|a_{kr}|$; we denote this number (and point) by a_{kr}, $k \geqslant 1$. Let the contour L agree with the imaginary axis, by bypassing the point a_{kr} from the left, and the point a_{-kr} from the right, $k \geqslant 1$. Then after expanding the integrals (1.5) in residue series at the poles $p = a_{kr}$, we have in accordance with the Mandel'shtam principle $P_{k1} < 0$ for $k = 1, 2, \ldots, S_1$ and $P_{k2} > 0$ for $-k = 1, 2, \ldots, S_2$.

Elimination of the function $C(p)$ from (1.6) results in a Wiener — Hopf equation

$$\eta^-(p) + \eta^+(p) = K(p)[\sigma^-(p) + \sigma^+(p)], \quad p \in L, \quad K(p) = N_1^{-1}(p) N_2(p) \qquad (1.8)$$

Let us construct the canonical solution of the homogeneous equation $\eta_0^-(p) = K(p)\sigma_0^+(p)$, $p \in L$. Let us apply the usual method [17, 8] of extracting the Riemann problem $\eta_2^-(p) = K_2(p)\sigma_2^+(p)$ with zero subscript on the imaginary axis from (1.8) by using trigonometric functions. Let us set

$$\sigma_0^+(p) = \sigma_1^+(p)\sigma_2^+(p), \quad K(p) = K_1(p) K_2(p) \qquad (1.9)$$

$$K_1(p) = \frac{D}{4G} p^3 \operatorname{ctg}^3 \pi p \prod_{k=1}^{S_1} \operatorname{ctg} \pi (p - a_{k1}) \operatorname{ctg} \pi (p + a_{k1}) \times$$

$$\left[\prod_{k=1}^{S_2} \operatorname{ctg} \pi (p - a_{k2}) \operatorname{ctg} \pi (p + a_{k2})\right]^{-1}$$

and let us use the formula

$$z \operatorname{ctg} \pi z = \Gamma(1 - z)\Gamma(1 + z)\Gamma^{-1}(1/2 - z)\Gamma^{-1} \times (1/2 + z).$$

Factoring the function $K_1(p)$ by elementary means in conformity with the Mandel shtam principle, we obtain

$$\sigma_1^+(p) = \frac{\Gamma^3(1/2 + p)}{\Gamma^3(1 + p)} \prod_{k=1}^{S_1} \frac{(p + a_{k1})\Gamma(1/2 + p - a_{k1})\Gamma(1/2 + p + a_{k1})}{\Gamma(1 + p - a_{k1})\Gamma(1 + p + a_{k1})} \times \tag{1.10}$$
$$\prod_{k=1}^{S_2} \frac{\Gamma(1 + p - a_{k2})\Gamma(1 + p + a_{k2})}{(p + a_{k2})\Gamma(1/2 + p - a_{k2})\Gamma(1/2 + p + a_{k2})}$$

According to (1.6) and (1.9), the function $K_2(p)$ satisfies the Hölder condition on the imaginary axis, $K_2(i\gamma) = 1 + 4GD^{-1}|\gamma|^{-3} + O(e^{-\gamma x})$ for $|\gamma| \to \infty$. We therefore have [18]

$$\sigma_2^+(p) = \operatorname{epx}\left\{\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\ln K_2(t)\,dt}{t - p}\right\} \tag{1.11}$$

The general solution of the inhomogeneous equation (1.8) is constructed taking into account the requirement of finiteness of the local energy under the beam edges, which is equivalent to the condition $u_5(x, 1) \sim Ax^{\varphi}$, $\varphi > -1$, $x \to +0$, or ([19], p. 48) $\sigma^+(p) \sim A\Gamma(\varphi + 1)\, p^{-\varphi-1}$ for $p \to \infty$. Since $\sigma_2^+(p) = O(1)$ and $\sigma_1^+(p) = O(p^{-3/2})$ for $p \to \infty$ according to (1.10) and (1.11), then $\varphi = -1/2$, and therefore [18]

$$\sigma^+(p) = \frac{\sigma_0^+(p)}{2\pi i}\int_L \left[\frac{\sigma^-(t)}{\sigma_0^+(t)} - \frac{\eta^+(t)}{K(t)\sigma_0^+(t)}\right]\frac{dt}{(t - p)} + \tag{1.12}$$
$$\sigma_0^+(p)\Pi(p), \quad \Pi(p) = Ap + B$$

where A and B are constants. However, (1.12) is hardly effective precisely because of the difficulties in calculating A, B for $\omega \neq 0$, and will not be used here.

Let us construct the solution in another form [2] which is well adapted for computations on the vibrational load of the foundation (strip) and the support beam slabs of a skeleton structure having a point contact with the slabs [20]. If $f_r(x) = f_r e^{ax}$ $(r = 1, 2)$, then we obtain by an elementary method, the Liouville theorem [19]

$$\sigma^+(p) + \sigma^-(p) = \sigma_0^+(p)\{f_1[\sigma_0^+(a)(a - p)]^{-1} + f_2[\eta_0^-(a)(p - a)]^{-1} + \Pi(p)\} \tag{1.13}$$

Let $f_1(x) \equiv 0$, $f_2(x) = Q_2\delta(x - c)$ in the problem (1.2), (1.3), where $\delta(x)$ is the Dirac delta function. We seek its solution u_s^{δ}, i.e., the Green's function, in the form of the sum of solutions of the fundamental problem, as in [2]:

$$u_s(x, 0) = u_2(x, 0) = u_3(x, 1), \quad \eta(x) = Q_2\delta(x - c) \tag{1.14}$$

evidently being expressed by the formulas (1.5), where

$$C(p) = Q_2 e^{-pc} N_2^{-1}(p) \tag{1.15}$$

and the solutions of the mixed (correcting) problem (1.2), (1.3), where

$$f_1(x) = \sum_{k=1}^{\infty} Q_2 e^{a_{k2}(x-c)} N_1(a_{k2}) [N_2'(a_{k2})]^{-1}, \quad f_2(x) \equiv 0 \tag{1.16}$$

The form of the function $f_1(x)$ permits using (1.13) in the solution of the problem (1.2), (1.3), (1.16). Consequently, taking account of (1.5) and (1.15) we obtain

$$u_s^\delta(x, y) = \frac{Q_2}{2\pi i}\int_L \left\{\frac{e^{-pc}}{N_2(p)} + \frac{1}{E_1(p)}\left[\sum_{n=1}^{\infty} t(a_{n2}, p) + \Pi(p)\right]\right\} U_s(p, y) e^{px} dp \tag{1.17}$$

$$t(\tau, p) = \frac{e^{-\tau c} E_1(\tau)}{(\tau - p) N_2'(\tau)}, \quad E_1(p) = \frac{N_1(p)}{\sigma_0^+(p)}$$

The Green's function of the problem of an overload is calculated analogously when $f_1(x) = Q_1\delta(x - c)$, $c < 0$, $f_2(x) \equiv 0$, $P_2 = P_3 = 0$ in (1.2) – (1.4).

The stress intensity factor under the beam edges is found by means of known asymptotic estimates [10] and has the form

$$u_5^\delta(x, 1) = A(\pi x)^{-1/2} + O(\sqrt{x}), \quad x \to +0$$

We write the expansion of the integrals (1.17) in a residue series to satisfy conditions (1.4) and the conditions at the endfaces of a rectangle

$$\lambda_1 \leqslant x \leqslant \lambda_2, \; 0 \leqslant y \leqslant 1, \; \lambda_1 < 0, \; \lambda_2 > c,$$

or a half-strip $\lambda_1 = -\infty$ and $\lambda_2 = \infty$ (see Sect. 3). Closing L on the right $(j = 1)$ and the left $(j = 2)$ by semicircles of radius $R_k = \pi\,|k|$, and taking account of the equality $E_1(p) = E_2(p) \equiv N_2(p)[\eta_0^-(p)]^{-1}$, $p \in L$, we obtain

$$u_s^\delta(\lambda_j, y) = (-1)^j Q_2 \sum_{k=1}^{\infty} t_j(-b_{kj}) U_s(-b_{kj}, y) e^{-b_{kj}l_j}, \; b_{kj} = (-1)^j a_{kj} \tag{1.18}$$

$$t_j(\tau) = (j-1)\frac{e^{-\tau c}}{N_2'(a_{k2})} + E_j^*(\tau)\left[\sum_{n=1}^{\infty} t(a_{n2}, \tau) + A\tau + B\right]$$

$(E_j^*(p) = \operatorname{res}[E_j(p)]^{-1})$

Let us evaluate the quantities $\psi_2(0)$, $\psi_3(0)$ by term-by-term differentiation of the series (1.18) for $j = s = 2$, $y = 1$, $\lambda_2 < c$.

Using the identity

$$DU_2(a_{k2}, 1) = N_1(a_{k2}) d_{k2}^{-1}, \quad d_{k2} = \alpha D^{-1} - a_{k2}^4 \tag{1.19}$$

we obtain a system of two equations in the complex unknowns A and B from condition (1.4), $(m = 2, 3)$

$$\sum_{k=-1}^{-\infty} \frac{N_1(a_{k2})\, a_{k2}{}^m}{N_2'(a_{k2})\, d_{k2}} \left\{ \eta_0^-(a_{k2}) \left[\sum_{n=1}^{\infty} t(a_{n2}, a_{k2}) + A a_{k2} + B \right] - \frac{(-1)^m}{e^{-a_{k2} c}} \right\} = \frac{P_m}{Q_2} \tag{1.20}$$

Because of the estimates (1. 7), the k-th terms of these series decrease not slower than $|k|^{-7/2}$ for $m = 2$ and $|k|^{-5/2}$ for $m = 3$, and according to (1. 17), the inner series converge exponentially in n.

Let us examine questions referring to the uniqueness of solutions in the form (1. 12) or (1. 17). In investigating the uniqueness, i. e., in computing the degree of the polynomial $\Pi(p)$, the representation (1. 10), (1. 11) imposed no constraints on either the residues of the function $[K(p)]^{\pm 1}$ at the poles $p = a_{kr}$, $k \leqslant S_r$, or on the mutual location of the points a_{k1} and a_{k2} on the imaginary axis. Each kr-th factor in the product (1. 10) tends to one for $|\operatorname{Im} p| \to \infty$, $\operatorname{Re} p \geqslant 0$. Hence, if any set of points $\{a_{k1}, a_{n2}\}$ ($\{a_{-k1}, a_{-n2}\}$) is bypassed on the right (left) say, for $k \leqslant S_1$, $n \leqslant S_2$, despite the Mandel'shtam principle, and the rest of the poles and zeroes on the imaginary axis on the left (right) afterwards, then the conditions of the generalized Liouville theorem governing the uniqueness and nonuniqueness do not change, as before $\sigma^+_1(p) \sim p^{-3/2}$ as $p \to \infty$. In particular, it hence follows that the Sommerfield principle also generates a unique solution in the form (1. 12), (1. 17), and if there are backward waves among those being propagated (bringing energy from infinity), then the magnitudes of the corresponding sources are not arbitrary at infinity but are determined in a unique manner by the functions $f_r(x)$ and the quantities P_r.

It is meanwhile necessary to note that the contour L does not certainly define the unique solution as holds in elastostatic boundary value problems. By replacement of the contour integrals by residue series at the poles a_{k1}, a_{k2}, the representation (1. 10), (1. 11) permits easily to establish for what selection of the contour L a nonunique solution can be obtained, how many arbitrary constants are and what the corresponding pattern of the wave process. For example, if the countour L selected according to the Mandel'shtam principle is displaced so that it would bypass any symmetric points $p = a_{k2}$ and $p = -a_{k2}$ $k = 1, 2, \ldots, S$, $S < S_2$, on the right, and its previous location is retained at the rest, then S factors $p - a_{k2}$ are added in the denominator of (1. 10) (the set of gamma functions in (1. 10) is not associated with the bypassing of the zeroes and poles on the imaginary axis, hence it is always single-valued). It hence follows that $\sigma_1^+(p) \sim p^{-3/2-S}$ as $p \to \infty$,and this means that according to the generalized Liouville theorem $S + 2$ arbitrary constants will enter into the solution. They can be determined by giving the amplitudes of S standing waves of the form $u_k(y) \cos(|a_{k2}|\, x) \times \cos \omega t$, which occur, according to (1. 12) for $x \to \infty$ after the mentioned shifting of the contour. Such an analysis is easily performed for an arbitrary bypassing of the points a_{kr} on the imaginary axis.

2. Let us construct a system of piecewise-homogeneous solutions (PHS) of the dynamic problem (1. 2) — (1. 4). The general form of its kr-th elements $u_{ks}{}^r(x, y)$, combined in two different subsystems ($r = 1, 2$), will be the same as in statics. Following [2], we obtain

$$u_{ks}^r(x, y) = C_{kr}U_s(b_{kr}, y)e^{b_{kr}x} + \frac{C_{kr}}{2\pi i}\int_L \left[\frac{E_{3-r}(b_{kr})}{(-1)^r(p-b_{kr})} + \Pi_{kr}\right] \times \tag{2.1}$$

$$\frac{U_s(p, y)\,dp}{E_r(p)\,e^{-px}}$$

$k = 1, 2, \ldots;\ \Pi_{kr} = A_{kr}p + B_{kr},$

where A_{kr}, B_{kr}, C_{kr} are arbitrary complex numbers. Analogously to the expansion (1.19), we have (δ_{rj} is the Kronecker delta)

$$u_{ks}^r(\lambda_j, y) = C_{kr}\left[\delta_{rj}U_s(b_{kr}, y)e^{b_{kr}\lambda_j} - (-1)^j\sum_{n=1}^{\infty} \times\right. \tag{2.2}$$

$$\left. T_{kn}^{rj}U_s(-b_{nj}, y)e^{-b_{nj}\lambda_j}\right]$$

$$T_{kn}^{rj} = [(-1)^r(b_{kr} + b_{nj})^{-1}E_{3-r}(b_{kr}) + A_{kr}b_{nj} - B_{kr}]E^*_j(-b_{nj})$$

The constants A_{kr}, B_{kr} are calculated from the conditions $\partial^m u_{k2}^r(x, y) / \partial x^m = 0,\ x = 0,\ y = 1,$ which yield a system of two equations for each A_{kr}, B_{kr}

$$(r-1)DU_2(a_{k2}, 1)a_{k2}^m + \sum_{n=-1}^{-\infty} T_{kn}^{r2}d_{n2}^{-1}a_{n2}^m N_1(a_{n2}) = 0 \tag{2.3}$$

$$(m = 2, 3;\ r = 1,2;\ k = 1, 2, \ldots)$$

after term-by-term differentiation of the series (2.2) for $s = j = 2$ and passage to the limit as $x = 0$.

The series converge in n, as do the external series (1.20).

The stresses generated by the kr - th PHS under the beam edges have the form

$$u_{k5}^r(x, 1) = A_{kr}(\pi x)^{-1/2} + O(\sqrt{x}),\quad x \to +0$$

Following [10], it can be shown that the functions $U_s(a_{kr}, y)$ the homogeneous solutions into which the elements of the PHS system are expanded in (2.2) for $x =$ const, will satisfy the general orthogonality relationship with the load

$$\int_0^1 [U_1(a_{kr}, y)U_4(a_{nr}, y) - U_3(a_{kr}, y)U_2(a_{nr}, y)]\,dy + \tag{2.4}$$

$$2(r-1)Da_{kr}(a_{nr}^2 + a_{kr}^2)U_2(a_{kr}, 1)U_2(a_{nr}, 1) = \delta_{kn}T_{kr}$$

This relationship also holds upon replacing a_{kr} by b_{kr}; it has been obtained in [10] for $r = 1$.

Diverse contact problems about the forced stationary vibrations of an arbitrary number of finite beams coupled to an elastic strip, half-strip, and rectangle, can be solved by using the PHS system (2.1). As analytic estimates and the results of computing the static strains of beams and stringers in combination with an elastic base [2– 9, 20], the solution hence turns out to be effective even in those cases when the strips or rectangles are comprised of homogeneous reinforced rectangles and half-strips with different elastic characteristics, i.e., when the base has a vertically laminar structure and the stiffness of the reinforcing elements are piecewise-continuous. To solve viscoelasticity problems it is sufficient to set $S_r = 0$ in (1.10) and to replace the real moduli of G and E_0 by complex moduli.

The PHS method is easily transferred to problems of plate bending vibrations, to mixed problems of stationary vibrations of cylindrical bodies [1, 3, 4, 7], to kindred problems of acoustic and electromagnetic wave diffraction. This method is not applicable to dynamic boundary value problems for wedgeshaped and conical domains.

Setting $D = \infty$, and using (2.1) there is possible reduce to normal systems of Poincaré — Koch, the problems of the vibrations of stamps, as well as their dual problems of arbitrary slits in a half-strip and rectangle, particularly, periodic and doubly-periodic problems on the stationary vibrations of an inhomogeneous strip, half-strip, and place weakened by longitudinal and transverse slits.

3. Let us turn to examples. Consider two problems a) and b) for the rectangle $\lambda_1 \leqslant x \leqslant \lambda_2$, $0 \leqslant y \leqslant 1$. Let there be "cross" conditions on its endfaces

$$u_s(\lambda_j, y) = g_{sj}(y), \quad \psi_m(\lambda_2) = F_m, \quad j = 1, 2 \tag{3.1}$$

in which a) $s = 1.3$, $m = 1.3$; b) $s = 2.4$, $m = 0.2$. The conditions (1.2) — (1.4) are conserved on the longitudinal sise, where $f_1(x) \equiv 0$, $f_2(x) = Q_2\delta(x - c)$; all the loads are cophasal.

We seek the solution of both problems in series of PHS

$$u_s(x, y) = u_s^\delta(x, y) + \sum_{r=1}^{2} \sum_{k=1}^{\infty} u_{ks}^r(x, y) \tag{3.2}$$

satisfying conditions (1.2) — (1.4) at once. To satisfy conditions (3.1), we substitute the series (3.2) in their left sides and then the expansions (1.18) and (2.2). We change the order of summation with respect to k and n in the double series being formed. We multiply both sides of conditions (3.1) successively a) by $U_4(b_{nj}, y)$, $-U_2(b_{nj}, y)$, $2b_{n2}^2U_2(b_{n2}, 1)$, $2U_2(b_{n2}, 1)$, and b) by $U_1(b_{nj}, y)$, $-U_3(b_{nj}, y)$, $2b_{n2}U_2(b_{n2}, 1)$, $2b_{n2}^3U_2(b_{n2}, 1)$. We add the four equalities manipulated in such a way and we integrate with respect to y between zero and one. Introducing the new unknown $X_{kr} = C_{kr}e^{b_{kr}\lambda_r}$, we obtain a normal Poincaré — Koch system from the total equation [23] because of the evenness of the relationship $U_s(-b_{kr}, y) = (-1)^s U_s(b_{kr}, y)$ and the orthogonality (2.4):

$$X_{kj}(-1)^{j+l} - \sum_{r=1}^{2} \sum_{n=1}^{\infty} X_{nr}T_{nk}^{rj} \exp(-b_{kj}\lambda_j - b_{nr}\lambda_r) = t_{kj}^{l} - Q_2 t_j(b_{kj})\, e^{-b_{kj}\lambda_j} \tag{3.3}$$

$$t_{kj}^1 = \frac{1}{T_{kj}}\left\{(-1)^{j+1}\int_0^1 [g_{1j}(y)\,U_4(b_{kj}, y) - g_{3j}(y)\,U_2(b_{kj}, y)]\,dy + \right.$$
$$\left. 2(j-1)(F_1a_{k2}^2 + F_3)\,U_2(b_{k2}, 1)\right\}$$

$$t_{kj}^2 = \frac{1}{T_{kj}}\left\{(-1)^{j}\int_0^1 [g_{4j}(y)\,U_1(b_{kj}, y) - g_{2j}(y)\,U_3(b_{kj}, y)]\,dy + \right.$$
$$\left. 2(j-1)(F_0a_{k2}^2 + F_2)\,a_{k2}U_2(a_{k2}, 1)\right\}$$

Here $j = 1,2$; $k = 1,2, \ldots$; $l = 1$, in problem a), and $l = 2$ in problem b).

The solution constructed for particular values of the given functions and geometric parameters will also become the solution of certain important contact problems for an elastic strip. If $g_{sj}(y) \equiv 0$ and $F_m = 0$, then conditions (3.1) define the problem of a periodic system, with period $2(\lambda_2 - \lambda_1)$ of beams of length $2\lambda_2$, coupled to a strip $0 \leqslant y \leqslant 1$. Applied to each beam are a) symmetric or b) skew-symmetric forces, the sum of the solutions of problems a) and b) corresponds to an arbitrary load. In contrast to periodic statics problems where the longitudinal strains of the strip can be controlled at infinity, say, here all the dynamic parameters are single-valued at infinity: the waves are not propagated through the rectangle endfaces, and the energy flux at the endfaces equal zero. The constraints imposed on the Saint-Venant principle in dynamics are hence seen. For instance, for sufficiently large ω, the solution of the periodic problem does not approach the solution of the problem of the vibration of an arbitrarily large but finite number of periodically arranged identical beams on a strip subjected on an identical load for any x, after the occurrence of propagating waves.

If we set $A_{k1} = 0, \lambda_1 = -\infty$ in the solution of (3.2), (2.1), (3.3) for $g_{sj}(y) \equiv 0, F_m = 0$, then it is converted into the Green's function of the problem of stationary vibrations of a beam of length $2\lambda_2$ on a strip $0 \leqslant y \leqslant 1$. Setting $A_{k2} = 0, \lambda_2 = \infty$, we obtain the solution of the problem of the vibration of two semi-infinite beams between whose ends the distance is $2\lambda_1$. The conditions at infinity in these problems correspond to the Mandel'shtam principle. Under other radiation conditions, the uniqueness and ambiguity of the solutions is determined by the selection of the contour L, as in Sect. 1.

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